# Kurven sehen und verstehen

Haftendorn Juli 2017, http://www.kurven-sehen-und-verstehen.de

# Inversiongleichungen, Ursprungs-Kreis, Radius k,

In[134]:= `Quit`

beende Kernel

In[171]:= $\text{invert} = \left\{x \to \frac{ki^2\,x}{x^2+y^2},\ y \to \frac{ki^2\,y}{x^2+y^2}\right\}$

Out[171]= $\left\{x \to \frac{ki^2\,x}{x^2+y^2},\ y \to \frac{ki^2\,y}{x^2+y^2}\right\}$

# Inversion der Cassini’schen Kurven

## Cassini’sche Kurven

In[23]:= `{e =., k =., ki =.};`

In[24]:= $\text{cassini} = \left(x^2+y^2\right)^2 - 2\,e^2\left(x^2-y^2\right) == k^4 - e^4$

Out[24]= $-2\,e^2\left(x^2-y^2\right) + \left(x^2+y^2\right)^2 == -e^4 + k^4$

In[175]:= `(cassini) /. invert // Simplify`

vereinfache

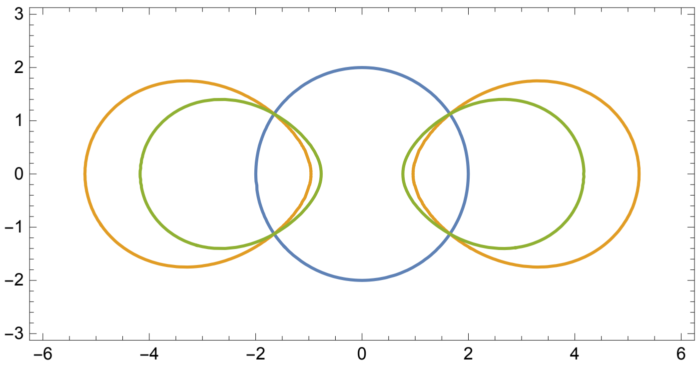
Out[175]= $e^4 + \frac{ki^8 + 2\,e^2\,ki^4\left(-x^2+y^2\right)}{\left(x^2+y^2\right)^2} == k^4$

## Das sind alle Kreis-Spiegelbilder  Cassini'schen Kurven in Mittelpunktslage

In[182]:=
```
{ki = 2, k = 2.9, e = 3};
ContourPlot[{x^2 + y^2 == ki^2, e^4 + (ki^8 + 2 e^2 ki^4 (-x^2 + y^2))/(x^2 + y^2)^2 == k^4,
  -2 e^2 (x^2 - y^2) + (x^2 + y^2)^2 == -e^4 + k^4}, {x, -6, 6}, {y, -3, 3}, AspectRatio → Automatic]
```

Konturgraphik; Seitenverhältnis; automatisch

Out[183]=

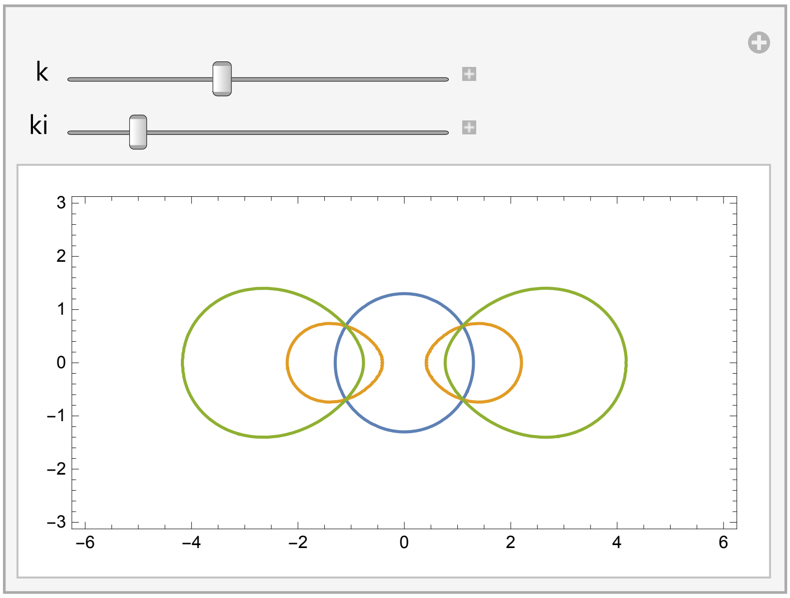


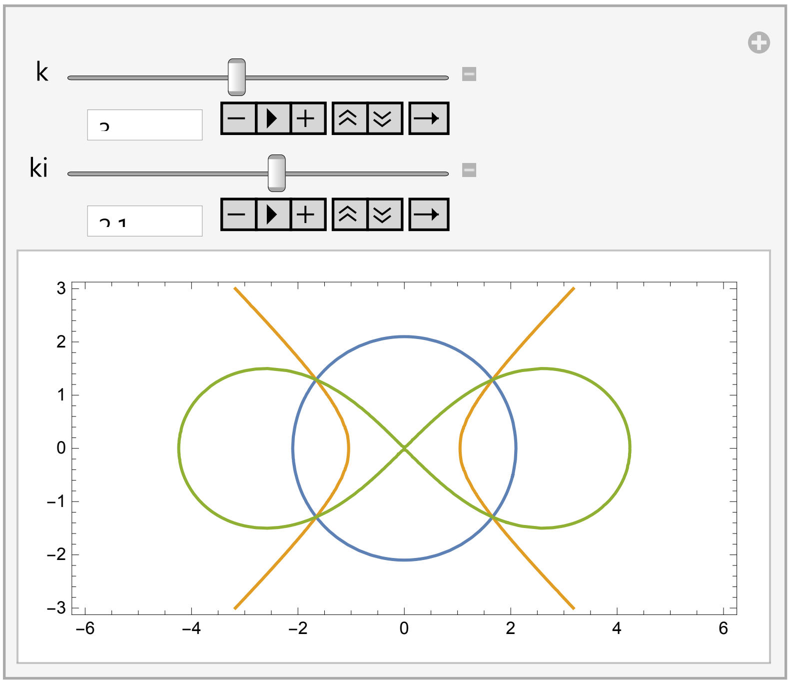

In[184]:=
```
{ki =., k =., e =.};
```

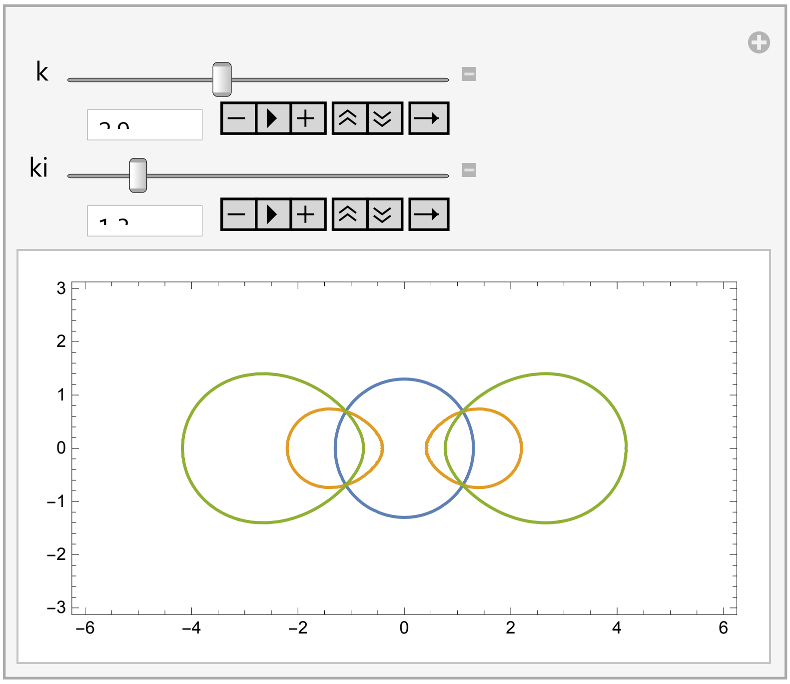
In[223]:=
```
e = 3;
Manipulate[ContourPlot[
  {x^2 + y^2 == ki^2, e^4 + (ki^8 + 2 e^2 ki^4 (-x^2 + y^2))/(x^2 + y^2)^2 == k^4, -2 e^2 (x^2 - y^2) + (x^2 + y^2)^2 == -e^4 + k^4},
  {x, -6, 6}, {y, -3, 3}, AspectRatio → Automatic],
 {{k, 2.9}, 2, 4.3, 0.1},
 {{ki, 1.3}, 1, 3, 0.1}, SaveDefinitions → True]
```

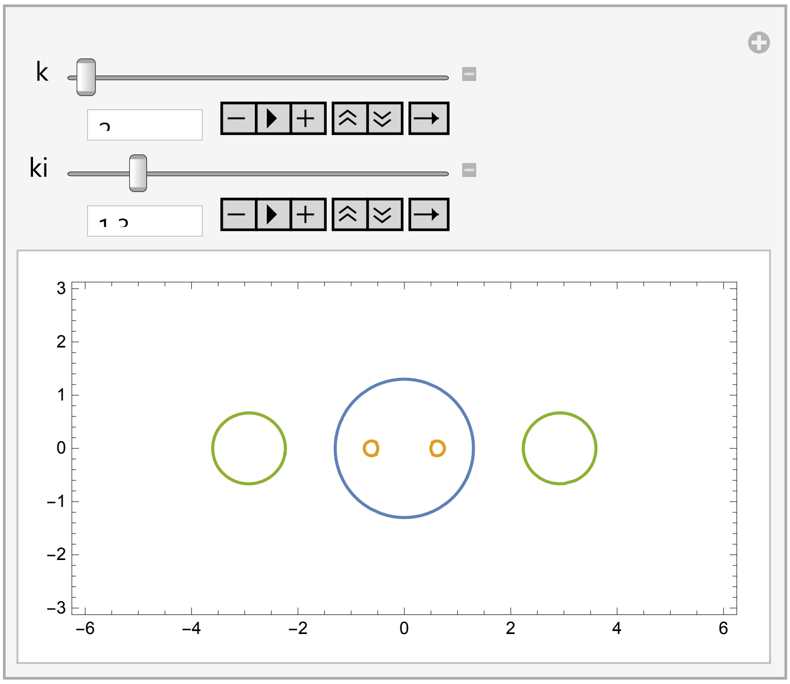
manipuliere; Konturgraphik; Seitenverhältnis; automatisch; speichere Definitionen; wahr

Out[223]=




k
ki
3
2
1
0
−1
−2
−3
−6
−4
−2
0
2
4
6


k
ki
3
2
1
0
−1
−2
−3
−6
−4
−2
0
2
4
6




Es scheint anallagmatische Konstellationen zu geben, bloß wie finde ich die?

---

# Suche anallagmatische Fälle

In[193]:= $$\text{casInv} = e^4 + \frac{ki^8 + 2\,e^2\,ki^4\,(-x^2 + y^2)}{(x^2 + y^2)^2} == k^4$$

$$\text{cassini} = -2\,e^2\,(x^2 - y^2) + (x^2 + y^2)^2 == -e^4 + k^4$$

Out[193]= $$e^4 + \frac{ki^8 + 2\,e^2\,ki^4\,(-x^2 + y^2)}{(x^2 + y^2)^2} == k^4$$

Out[194]= $$-2\,e^2\,(x^2 - y^2) + (x^2 + y^2)^2 == -e^4 + k^4$$

Ich berechne die Scheitelpunkte

In[200]:= `Solve[cassini /. y → 0, x]`
löse
`Solve[casInv /. y → 0, x]`
löse

Out[200]= $$\left\{\left\{x \to -\sqrt{e^2 - k^2}\right\}, \left\{x \to \sqrt{e^2 - k^2}\right\}, \left\{x \to -\sqrt{e^2 + k^2}\right\}, \left\{x \to \sqrt{e^2 + k^2}\right\}\right\}$$

Out[201]= $$\left\{\left\{x \to -\frac{ki^2}{\sqrt{e^2 - k^2}}\right\}, \left\{x \to \frac{ki^2}{\sqrt{e^2 - k^2}}\right\}, \left\{x \to -\frac{ki^2}{\sqrt{e^2 + k^2}}\right\}, \left\{x \to \frac{ki^2}{\sqrt{e^2 + k^2}}\right\}\right\}$$

**Wenn dass dieselben Scheitel sein sollen, muss gelten**

In[213]:= $$e =.;\ \text{Solve}\left[\frac{ki^2}{\sqrt{e^2 - k^2}} == \sqrt{e^2 + k^2},\ ki\right]$$
löse

In[214]:= $\{\{ki \to -(e^2-k^2)^{1/4}(e^2+k^2)^{1/4}\}, \{ki \to (e^2-k^2)^{1/4}(e^2+k^2)^{1/4}\}\}$ /. {e → 3, k → 2.9}

Out[214]= {{ki → -1.79025}, {ki → 1.79025}}

Das ist die Kombination, die ich von Hand gefunden hatte. Bei näherem Hinsehen

In[221]:= $ki^4 == e^4 - k^4$;

Der Fall mit k>e

In[216]:= Solve[cassini /. y → 0, x]
löse
Solve[casInv /. x → 0, y]
löse

Out[216]= $\{\{x \to -\sqrt{e^2-k^2}\}, \{x \to \sqrt{e^2-k^2}\}, \{x \to -\sqrt{e^2+k^2}\}, \{x \to \sqrt{e^2+k^2}\}\}$

Out[217]= $\{\{y \to -\frac{ki^2}{\sqrt{-e^2-k^2}}\}, \{y \to \frac{ki^2}{\sqrt{-e^2-k^2}}\}, \{y \to -\frac{ki^2}{\sqrt{-e^2+k^2}}\}, \{y \to \frac{ki^2}{\sqrt{-e^2+k^2}}\}\}$

In[218]:= e =.; Solve$\left[\frac{ki^2}{\sqrt{-e^2-k^2}} == \sqrt{-e^2+k^2}, ki\right]$
löse

Out[218]= $\{\{ki \to -(-e^2-k^2)^{1/4}(-e^2+k^2)^{1/4}\}, \{ki \to (-e^2-k^2)^{1/4}(-e^2+k^2)^{1/4}\}\}$

bei näherem Hinsehen für den Fall k>e, die eingeschnürten Cassini’schen Kurve}

In[248]:= {e =., k =., ki =.};

In[249]:= $ki^4 == k^4 - e^4$;

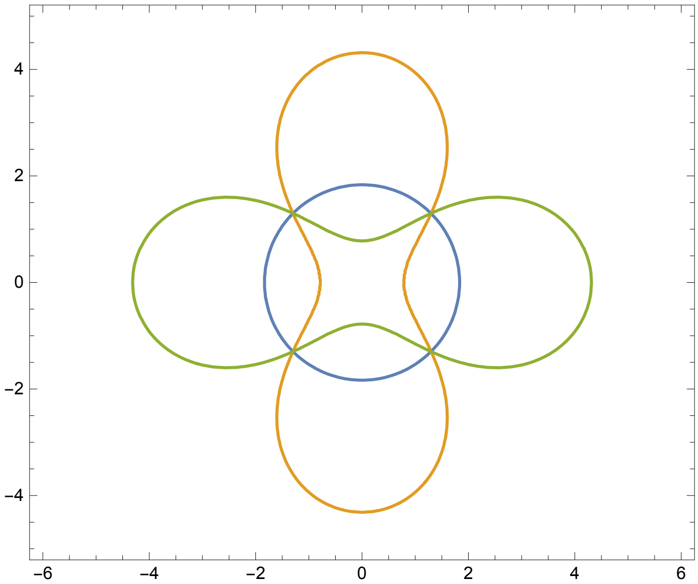
In[251]:= kiana = $(k^4 - e^4)^{\frac{1}{4}}$;

In[252]:= {e = 3, k = 3.1, ki = kiana};

$$\text{ContourPlot}\left[\left\{x^2+y^2==ki^2,\ e^4+\frac{ki^8+2\,e^2\,ki^4\left(-x^2+y^2\right)}{\left(x^2+y^2\right)^2}==k^4,\right.\right.$$
Konturgraphik

$$\left.\left.-2\,e^2\left(x^2-y^2\right)+\left(x^2+y^2\right)^2==-e^4+k^4\right\},\ \{x,-6,6\},\ \{y,-5,5\},\ \text{AspectRatio}\to\text{Automatic}\right]$$
Seitenverhältnis automatisch

Out[252]=




## bei näherem Hinsehen für den Fall e>k, die geteilten Cassini'schen Kurven

In[237]:= {ki =., k =., e =.};

In[239]:= $ki^4 == -k^4 + e^4$;

In[15]:= $kiana = \left(e^4 - k^4\right)^{\frac{1}{4}}$;

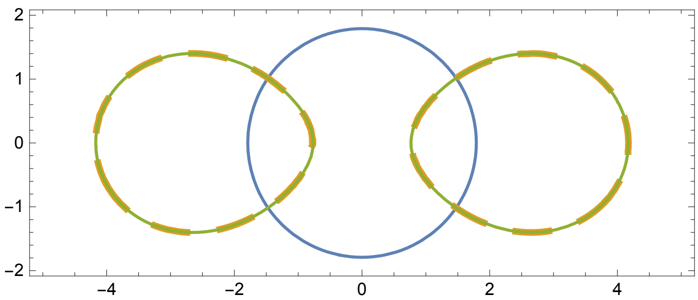
In[20]:= {e = 3, k = 2.9, ki = kiana};

$$\text{ContourPlot}\left[\left\{x^2+y^2==kiana^2,\ \frac{ki^4-2\,e^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}==-1,\right.\right.$$
Konturgraphik

$$\left.-2\,e^2\left(x^2-y^2\right)+\left(x^2+y^2\right)^2==-e^4+k^4\right\},\ \{x,-5,5\},\ \{y,-2,2\},\ \text{AspectRatio}\to\text{Automatic},$$
Seitenverhältnis automatisch

ContourStyle → {Dashing[{}], {Thickness[0.01], Dashing[0.05]}, Dashing[{}]}]
Konturenstil Strichelung Dicke Strichelung Strichelung

Out[20]=



# Cassini'sche Kurven,  Polare Form

In[51]:= 
```
CasPol = r^4 - 2 e^2 r^2 Cos[2 θ] == k^4 - e^4
```
Kosinus

Out[51]= $r^4 - 2 e^2 r^2 \mathrm{Cos}[2\theta] == -e^4 + k^4$

In[50]:= 
```
e =.
```

In[52]:= 
```
Solve[CasPol, r]
```
löse

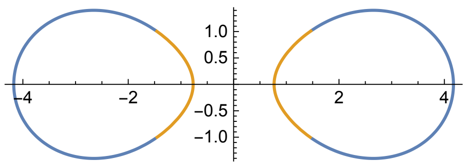
Out[52]= $$\left\{\left\{r \to -\sqrt{e^2 \mathrm{Cos}[2\theta] + \frac{\sqrt{-e^4 + 2k^4 + e^4 \mathrm{Cos}[4\theta]}}{\sqrt{2}}}\right\}, \left\{r \to \sqrt{e^2 \mathrm{Cos}[2\theta] + \frac{\sqrt{-e^4 + 2k^4 + e^4 \mathrm{Cos}[4\theta]}}{\sqrt{2}}}\right\}, \left\{r \to -\frac{\sqrt{2 e^2 \mathrm{Cos}[2\theta] - \sqrt{2}\sqrt{-e^4 + 2k^4 + e^4 \mathrm{Cos}[4\theta]}}}{\sqrt{2}}\right\}, \left\{r \to \frac{\sqrt{2 e^2 \mathrm{Cos}[2\theta] - \sqrt{2}\sqrt{-e^4 + 2k^4 + e^4 \mathrm{Cos}[4\theta]}}}{\sqrt{2}}\right\}\right\}$$

## Dieses sind die Polargleichung von Cassini'schen Kurven

$$\{e = 3, k = 2.9, ki = 2\}; \ \mathrm{PolarPlot}\left[\left\{\sqrt{e^2 \mathrm{Cos}[2\theta] + \frac{\sqrt{-e^4 + 2k^4 + e^4 \mathrm{Cos}[4\theta]}}{\sqrt{2}}}, \frac{\sqrt{2 e^2 \mathrm{Cos}[2\theta] - \sqrt{2}\sqrt{-e^4 + 2k^4 + e^4 \mathrm{Cos}[4\theta]}}}{\sqrt{2}}\right\}, \{\theta, 0, 2 \mathrm{Pi}\}\right]$$

Polardarstellung, Kreiszahl π

```
e =.;
k =.;
ki =.;  (* die Lösungen mit dem negativen Vorzeichen bringen keine anderen Kurven,
blau 1. Term,ocker 2. Term*)
```

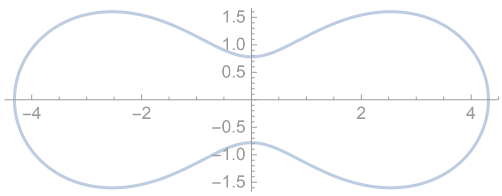
Out[156]= 


```
{e = 3, k = 3.1, ki = 2};
PolarPlot[{Sqrt[e^2 Cos[2 θ] + Sqrt[-e^4 + 2 k^4 + e^4 Cos[4 θ]]/Sqrt[2]]}, {θ, 0, 2 Pi}]
```

Polardarstellung Kreisza

```
e =.;
k =.;  (*für größere k , k>3=e) ist der 2. Term nicht definiert,
der erste gibt alles wider.*)
```

Out[103]=

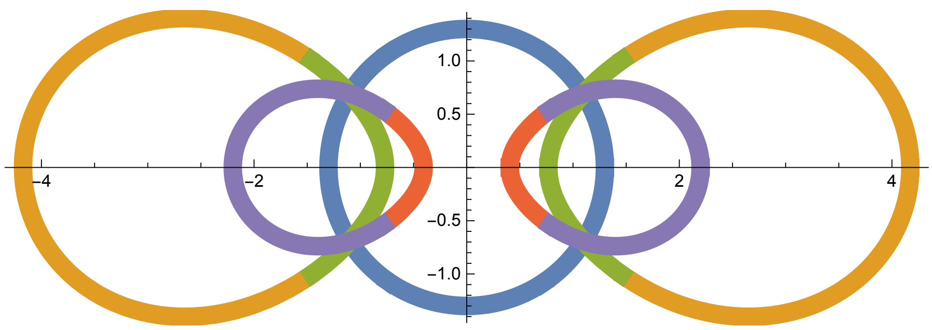



```
{e = 3, k = 2.9, ki = 1.3}; PolarPlot[{ki, Sqrt[e^2 Cos[2 θ] + Sqrt[-e^4 + 2 k^4 + e^4 Cos[4 θ]]/Sqrt[2]],
  Sqrt[2 e^2 Cos[2 θ] - Sqrt[2] Sqrt[-e^4 + 2 k^4 + e^4 Cos[4 θ]]]/Sqrt[2],
  ki^2 (Sqrt[e^2 Cos[2 θ] + Sqrt[-e^4 + 2 k^4 + e^4 Cos[4 θ]]/Sqrt[2]])^-1,
  ki^2 (Sqrt[2 e^2 Cos[2 θ] - Sqrt[2] Sqrt[-e^4 + 2 k^4 + e^4 Cos[4 θ]]]/Sqrt[2])^-1},
 {θ, 0, 2 Pi}, PlotStyle → Thickness[0.02]]
```

Polardarstellung Krei⋯ Darstellungsstil Dicke

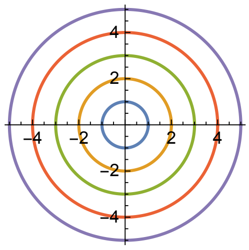
```
e =.;
k =.;
ki =.;
```

Out[39]=



In[113]:=
```
PolarPlot[{1, 2, 3, 4, 5}, {t, 0, 2 π}]
```
Polardarstellung

Out[113]=



In[160]:= 

```
{e = 3, ki = 1.3};
```

$$\text{Manipulate}\Big[\text{PolarPlot}\Big[\Big\{\sqrt{e^2\,\text{Cos}[2\,\theta] + \frac{\sqrt{-e^4 + 2\,k^4 + e^4\,\text{Cos}[4\,\theta]}}{\sqrt{2}}},$$

$$\frac{\sqrt{2\,e^2\,\text{Cos}[2\,\theta] - \sqrt{2}\,\sqrt{-e^4 + 2\,k^4 + e^4\,\text{Cos}[4\,\theta]}}}{\sqrt{2}}, \text{ki},$$

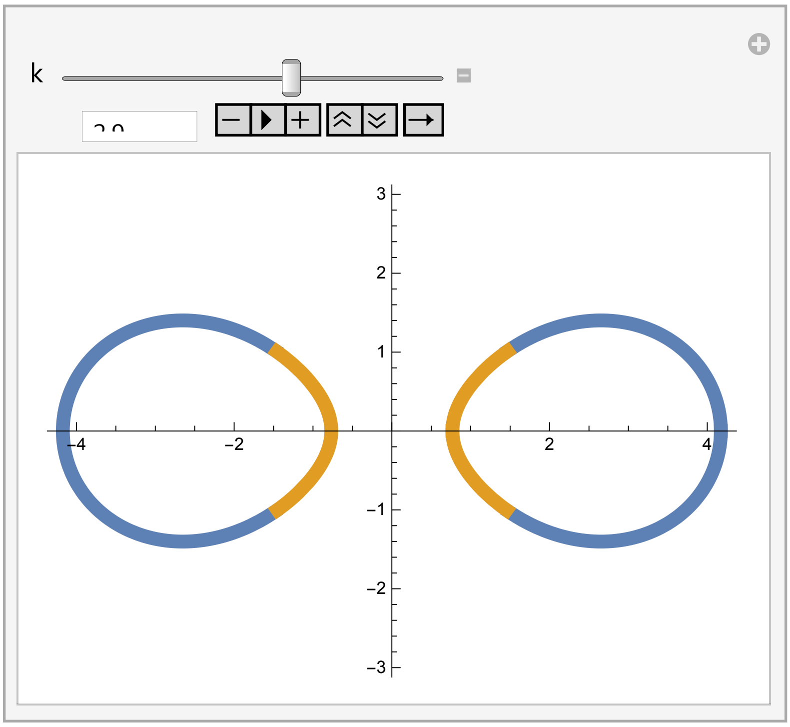
$$\text{ki}^2\left(\sqrt{e^2\,\text{Cos}[2\,\theta] + \frac{\sqrt{-e^4 + 2\,k^4 + e^4\,\text{Cos}[4\,\theta]}}{\sqrt{2}}}\right)^{-1},$$

$$\text{ki}^2\left(\frac{\sqrt{2\,e^2\,\text{Cos}[2\,\theta] - \sqrt{2}\,\sqrt{-e^4 + 2\,k^4 + e^4\,\text{Cos}[4\,\theta]}}}{\sqrt{2}}\right)^{-1}\Big\},$$

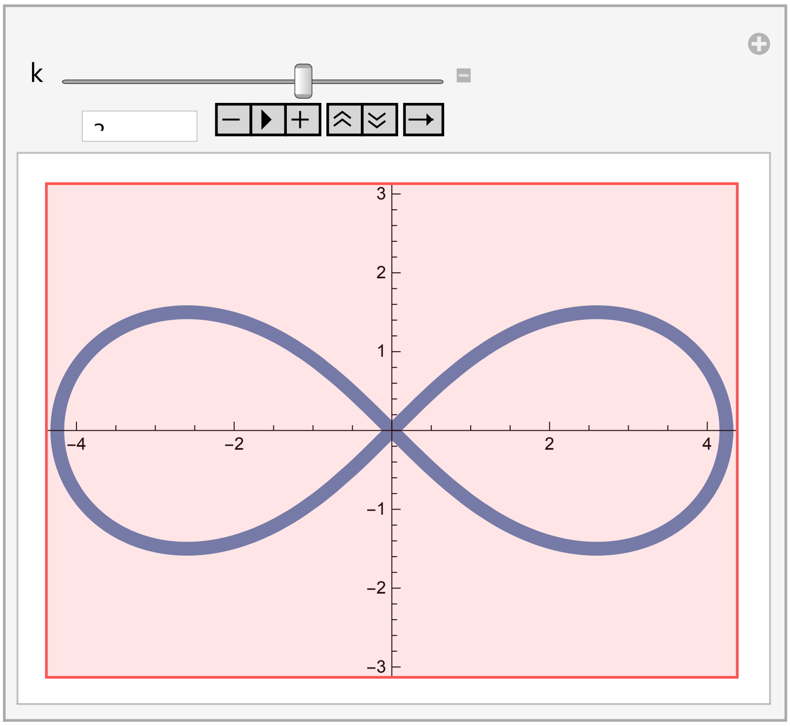
```
  {θ, 0, 2 Pi},
  PlotRange → {{-4.2, 4.2}, {-3, 3}}, PlotStyle → Thickness[0.02]],
 {{k, 2.9}, 1.2, 4, 0.1}
]
```

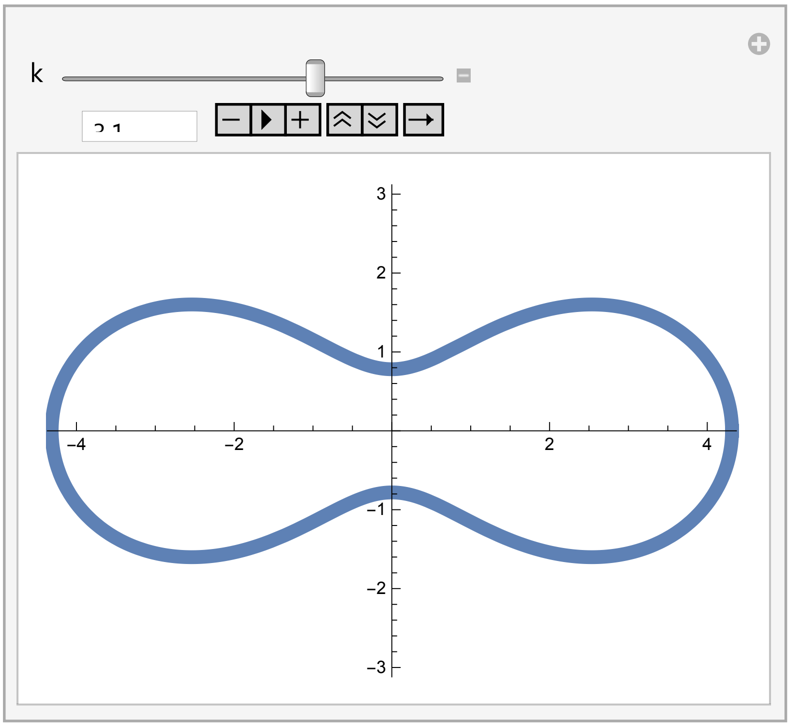
manipuliere | Polardarstellung | Kreiszahl π | Koordinatenbereich der Graphik | Darstellungsstil | Dicke

Out[161]= 



In[162]:=




## Für welches k haben die einteiligen Cassini’schen Kurven keine Einbuchtung mehr?

Strategie: Gerade durch den obern Scheitel, wann schneidet diese Gerade nicht nochmal?

### oberer Scheitel der blauen Cassini'schen Kurven

In[25]:= `cassini`

Out[25]= $-2\,e^2\,(x^2-y^2)+(x^2+y^2)^2 == -e^4+k^4$

In[27]:= `Solve[(-2 e^2 (x^2 - y^2) + (x^2 + y^2)^2 == -e^4 + k^4) /. x → 0, {y}]`
löse

Out[27]= $\{\{y\to-\sqrt{-e^2-k^2}\},\{y\to\sqrt{-e^2-k^2}\},\{y\to-\sqrt{-e^2+k^2}\},\{y\to\sqrt{-e^2+k^2}\}\}$

In[30]:= `cassini /. {y → √(-e^2 + k^2)}`

Out[30]= $-2\,e^2\,(e^2-k^2+x^2)+(-e^2+k^2+x^2)^2 == -e^4+k^4$

In[31]:= `Solve[-2 e^2 (e^2 - k^2 + x^2) + (-e^2 + k^2 + x^2)^2 == -e^4 + k^4, {x}]`
löse

Out[31]= $\{\{x\to 0\},\{x\to 0\},\{x\to-\sqrt{2}\,\sqrt{2\,e^2-k^2}\},\{x\to\sqrt{2}\,\sqrt{2\,e^2-k^2}\}\}$

---

# Für welches k fällt der eine Scheitel der zweiteiligen Kurven auf den Rand des Inversionskreises?

### x-Achsen-Scheitel der grünen Cassini'schen Kurven

In[32]:= `cassini`

In[37]:= $-2\,e^2\,(x^2-y^2)+(x^2+y^2)^2 == -e^4+k^4$

Out[37]= $-2\,e^2\,(x^2-y^2)+(x^2+y^2)^2 == -e^4+k^4$

In[38]:= `Solve[(-2 e^2 (x^2 - y^2) + (x^2 + y^2)^2 == -e^4 + k^4) /. y → 0, {x}]`
löse

Out[38]= $\{\{x\to-\sqrt{e^2-k^2}\},\{x\to\sqrt{e^2-k^2}\},\{x\to-\sqrt{e^2+k^2}\},\{x\to\sqrt{e^2+k^2}\}\}$

In[41]:= $\sqrt{5.}$

Out[41]= 2.23607